

THE CP-MATRIX COMPLETION PROBLEM

ANWA ZHOU AND JINYAN FAN

ABSTRACT. A symmetric $n \times n$ matrix A is called completely positive (CP) if there exists an entrywise nonnegative matrix $B \in \mathbb{R}^{n \times m}$ such that $A = BB^T$. Given a nonnegative E -matrix whose entries are partially given, the CP -completion problem is to decide whether we can assign values to the ungiven entries such that the completed matrix is CP . The main results of this paper are as follows: 1) We first present an SDP algorithm to solve the CP -completion problem. 2) When all the diagonal entries of the E -matrix are given, the algorithm can give a certificate for the non- CP -completeness, and is almost always guaranteed to get a CP -completion if it is CP -completable. 3) When the diagonal entries of the E -matrix are partially given, if the corresponding maximal principal submatrix is not CP -completable, then a certificate can be obtained; if the maximal principal submatrix is CP -completable, then the algorithm is almost always guaranteed to get a CP -completion. In particular, if the E -matrix has only one diagonal given, then it can always be completed to a CP -matrix. In all our computational experiments, we always get a CP -completion if it exists.

1. INTRODUCTION

A real $n \times n$ symmetric matrix A is called *completely positive* (CP) if there exist nonnegative vectors u_1, \dots, u_m such that

$$(1.1) \quad A = u_1 u_1^T + \dots + u_m u_m^T,$$

where the number m is called the *length of the factorization* (1.1). The smallest possible m is called the CP -rank of A . If A is CP , we call (1.1) the CP -decomposition of A . Equivalently, A is CP if it could be factorized as $A = BB^T$, where $B \in \mathbb{R}^{n \times m}$ is an entrywise nonnegative matrix. Hence a CP -matrix is not only positive semi-definite but also nonnegative.

CP -matrices have wide applications in mixed binary quadratic programming [7], approximating stability numbers [10], the MaxClique problem [29, 33], single quadratic constraint problem [30], standard quadratic optimization [3], general quadratic programming [31], and so on. Some NP-hard problems can also be reformulated as linear optimization problems over the cone of CP -matrices [16, 21, 23]. These important applications motivate researchers to study whether a matrix is CP or not. The interested reader may refer to [1, 2, 4–6, 8, 11–13] for more details.

An interesting problem related to the CP -decomposition is the CP -completion problem. Let $E = \{(i_1, j_1), \dots, (i_l, j_l) \mid 1 \leq i_k \leq j_k \leq n, k = 1, \dots, l\}$. A real $n \times n$ symmetric partial matrix is called an E -matrix, if its entries are given for

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all $(i, j) \in E$, while all the other entries are ungiven. The *CP-completion problem* is to study whether we can assign values to the ungiven entries of the E -matrix such that the completed matrix is *CP*. As a symmetric matrix can be identified by a vector that consists of all its upper triangular entries, we take the given upper triangular entries to form the identifying vector of an E -matrix as follows:

$$\mathbf{a} \in \mathbb{R}^E,$$

where \mathbb{R}^E denotes the space of vectors that are indexed by $(i, j) \in E$. It is obvious that an E -matrix with an identifying vector $\mathbf{a} \in \mathbb{R}^E$ is *CP-completable* if and only if there exists a *CP*-matrix A such that $\mathbf{a} = A|_E$, where $A|_E$ denotes the E -truncated vector of the matrix A . We say an E -matrix is *non-CP-completable* if it could not be completed to a *CP*-matrix.

For example, consider the E -matrix [1, Example 2.23]:

$$(1.2) \quad \begin{bmatrix} 2 & 3 & 0 & * \\ 3 & 6 & 3 & 0 \\ 0 & 3 & 6 & 3 \\ * & 0 & 3 & 2 \end{bmatrix},$$

where $*$ denotes the ungiven entries of the E -matrix throughout this paper. The index set of (1.2) is $E = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$ and the corresponding identifying vector is $\mathbf{a} = (2, 3, 0, 6, 3, 0, 6, 3, 2)^T$. Note that for any nonnegative $*$ in the $(1, 4)$ -th position, the determinant of (1.2) is $\det = -27(* + 1) < 0$, so it is not *CP*-completable.

If we consider the E -matrix:

$$(1.3) \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & * \end{bmatrix}.$$

Then the index set is $E = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3)\}$ and the corresponding identifying vector is $\mathbf{a} = (1, 1, 1, 1, 1)^T$. Since

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$$

is completely positive and $\mathbf{a} = A|_E$, (1.3) is *CP*-completable.

Note that if the diagonal of a *CP*-matrix is zero, then all the off-diagonal entries on this row and column will be zeros too. Without loss of generality, we assume that the E -matrix we wish to analyze is nonnegative and all its given diagonals are strictly positive.

There are some results on *CP*-completion, which are related to the underlying graph of the E -matrix with the special structure. An E -matrix is called a *partial CP-matrix* if all its fully specified principal submatrices are *CP*. Every partial *CP*-matrix realization of a connected graph G is *CP*-completable if and only if G is a block-clique graph [1, 14]. If G is the specification graph of the E -matrix, we call the E -matrix a *G-partial matrix*. For a partial *CP*-matrix whose specification graph contains cycles, under some additional cycle conditions, the G -partial *CP*-matrix is *CP*-completable if and only if all the blocks of G are cliques or cycles [15].

The above *CP*-completion results are focused on the special E -matrix that have all diagonals given or the corresponding specification graph satisfies some structures. In this paper, we will study the *CP*-completion problem for the general

E -matrix. We will discuss the problem in three cases regarding whether the diagonals of the E -matrix are given or not. The first case is that all diagonals are given, and the other two cases are that all diagonals are ungiven and the diagonals are partially given. We will explore if an E -matrix can not be completed to a CP -matrix, how can we get a certificate for that? If an E -matrix is CP -completable, how can we get a CP -completion and give a CP -decomposition for this completion?

The paper is organized as follows. In Section 2, we analyze the CP -completion problem for the E -matrix with all diagonals given. We show that the CP -completion problem can be formulated as an \mathcal{E} -truncated Δ -moment problem (\mathcal{E} -T Δ MP) with some special \mathcal{E} and Δ . We present an SDP algorithm for the \mathcal{E} -T Δ MP; some properties of the algorithm are obtained. In Section 3, we discuss the CP -completion problem for the E -matrix with the diagonals ungiven and partially given, respectively. Some experimental results are given in Section 4. Finally, we conclude the paper in Section 5.

2. AN SDP ALGORITHM FOR CP-COMPLETION

Recently, Nie [24] presented a new efficient SDP algorithm for the \mathcal{E} -truncated Δ -moment problem (\mathcal{E} -T Δ MP), which is a generalization of the semidefinite approach proposed in [18] for the truncated Δ -moment problem (T Δ MP). In this section, we will show how to formulate the CP -completion problem as an \mathcal{E} -T Δ MP with a special \mathcal{E} and a compact set Δ . We then present an SDP algorithm for it, which is a special case of Nie's algorithm. Some basic properties of the algorithm will also be studied.

2.1. Formulation as \mathcal{E} -T Δ MPs. We begin with a sufficient and necessary condition for an E -matrix to be CP -completable. Let

$$(2.1) \quad \Delta = \{x \in \mathbb{R}^n : h(x) = 0, g(x) \geq 0\}$$

be a standard simplex, where $h = x_1 + \cdots + x_n - 1$ and $g = (x_1, \dots, x_n)$ are two tuples of polynomials. It is easy to see that an E -matrix with the identifying vector $\mathbf{a} \in \mathbb{R}^E$ is CP -completable if and only if there exist nonnegative vectors $v_1, \dots, v_m \in \Delta$ and $\rho_1, \dots, \rho_m > 0$ such that

$$(2.2) \quad A := \rho_1 v_1 v_1^T + \cdots + \rho_m v_m v_m^T \quad \text{and} \quad \mathbf{a} = A|_E.$$

Denote the set of nonnegative integers by \mathbb{N} . For any $\alpha := (\alpha_1, \dots, \alpha_n)^T \in \mathbb{N}^n$, we define $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Let

$$(2.3) \quad \mathcal{E} = \{\alpha \in \mathbb{N}^n : |\alpha| = 2, \varphi(\alpha) \in E\},$$

where $\varphi(\alpha)$ is the following one-to-one mapping from the index vector set $\{\alpha \in \mathbb{N}^n : |\alpha| = 2\}$ to the point pair set $\{(i, j) : 1 \leq i \leq j \leq n\}$:

$$\alpha = (0, \dots, 0, 1_i, 0, \dots, 0, 1_j, 0, \dots, 0)^T \mapsto (i, j),$$

or

$$\alpha = (0, \dots, 0, 2_i, 0, \dots, 0)^T \mapsto (i, i),$$

with 1_i and 2_i meaning the i -th entry of the vector is 1 and 2 respectively. We say \mathcal{E} is the index set of E . For example, if $n = 3$ and $E = \{(1, 2), (2, 2), (2, 3)\}$, then $\mathcal{E} = \{(1, 1, 0), (0, 2, 0), (0, 1, 1)\}$. The degree of \mathcal{E} is defined as $\deg(\mathcal{E}) := \max\{|\alpha| :$

$\alpha \in \mathcal{E}$. Thus $\deg(\mathcal{E}) \equiv 2$ for any E . Hence, with a proper term order, we could rewrite the identifying vector \mathbf{a} as

$$(2.4) \quad \mathbf{a} = (\mathbf{a}_\alpha)_{\alpha \in \mathcal{E}} \in \mathbb{R}^\mathcal{E},$$

where \mathbf{a}_α denotes the entry indexed by α and $\mathbb{R}^\mathcal{E}$ denotes the space of real vectors indexed by elements in \mathcal{E} . We call \mathbf{a} an \mathcal{E} -truncated moment sequence (\mathcal{E} -tms).

Let Δ be as in (2.1). The \mathcal{E} -truncated Δ -moment problem (\mathcal{E} -T Δ MP) studies whether or not a given \mathcal{E} -tms \mathbf{a} admits a Δ -measure μ , i.e., a nonnegative Borel measure μ supported in Δ such that $\mathbf{a}_\alpha = \int_\Delta x^\alpha d\mu$ for any $\alpha \in \mathcal{E}$, where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. A measure μ satisfying the above condition is called a Δ -representing measure for \mathbf{a} . Furthermore, we say a measure is *finitely atomic* if its support is a finite set, and is *m-atomic* if its support consists of at most m distinct points. In other words, if an \mathcal{E} -tms \mathbf{a} admits a m -atomic Δ -measure, then there exist $v_1, \dots, v_m \in \Delta$ and $\rho_1, \dots, \rho_m > 0$ such that $\mathbf{a} = \rho_1[v_1]_\mathcal{E} + \cdots + \rho_m[v_m]_\mathcal{E}$, where $[v_i]_\mathcal{E} := (v_i^\alpha)_{\alpha \in \mathcal{E}}$ for $i = 1, \dots, m$. Hence by (2.2), an E -matrix with the identifying vector $\mathbf{a} \in \mathbb{R}^\mathcal{E}$ is CP -completable if and only if \mathbf{a} admits a m -atomic Δ -measure, i.e.,

$$(2.5) \quad \mathbf{a} = \rho_1[v_1]_\mathcal{E} + \cdots + \rho_m[v_m]_\mathcal{E},$$

where $v_i \in \Delta$ and $\rho_i > 0$ ($i = 1, \dots, m$). Now we have formulated the CP -completion problem as an \mathcal{E} -T Δ MP with \mathcal{E} and Δ given by (2.3) and (2.1) respectively.

2.2. A semidefinite algorithm. We first review some basic conceptions. Denote $\mathbb{R}[x]_\mathcal{E} := \text{span}\{x^\alpha : \alpha \in \mathcal{E}\}$. $\mathbb{R}[x]_\mathcal{E}$ is said to be Δ -full if there exists a polynomial $p(x) \in \mathbb{R}[x]_\mathcal{E}$ such that $p(x)|_\Delta > 0$, which implies that $p(x)$ is always positive on Δ [28]. Denote $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}$ and $\mathbb{R}[x]_d := \text{span}\{x^\alpha : \alpha \in \mathbb{N}_d^n\}$. The dimension of $\mathbb{R}[x]_d$ is $s(d) = C_{n+d}^d = \frac{(n+d)!}{n!d!}$ [22]. For a tms $z \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ and $q \in \mathbb{R}[x]_{2k}$, a symmetric matrix $L_q^{(k)}(z)$ is called the k -th localizing matrix of q generated by z if it is linear in z and satisfies the Riesz function. The order of $L_q^{(k)}(z)$ is $s(k - \lceil \deg(q)/2 \rceil)$, where $\lceil t \rceil$ is the smallest integer that is not smaller than $t \in \mathbb{R}$ [24]. Especially, $L_1^{(k)}(z)$ is called the k -th order moment matrix and denoted by $M_k(z)$.

Let $g_0 = 1$ and $g_{n+1} := 1 - \|x\|_2^2$. Since the set Δ defined by (2.1) satisfies $\Delta \subseteq B(0, 1) := \{x \in \mathbb{R}^n : \|x\|_2^2 \leq 1\}$, it can be equivalently described as $\Delta = \{x \in \mathbb{R}^n : h = 0, g_B := (g_0, g_1, \dots, g_n, g_{n+1}) \geq 0\}$. If $n = 2$ and $k = 2$, then the k -th localizing matrices of polynomials h and g_B could be computed as follows:

$$L_1^{(2)}(z) = M_2(z) = \begin{bmatrix} z_{(0,0)} & z_{(1,0)} & z_{(0,1)} & z_{(2,0)} & z_{(1,1)} & z_{(0,2)} \\ z_{(1,0)} & z_{(2,0)} & z_{(1,1)} & z_{(3,0)} & z_{(2,1)} & z_{(1,2)} \\ z_{(0,1)} & z_{(1,1)} & z_{(0,2)} & z_{(2,1)} & z_{(1,2)} & z_{(0,3)} \\ z_{(2,0)} & z_{(3,0)} & z_{(2,1)} & z_{(4,0)} & z_{(3,1)} & z_{(2,2)} \\ z_{(1,1)} & z_{(2,1)} & z_{(1,2)} & z_{(3,1)} & z_{(2,2)} & z_{(1,3)} \\ z_{(0,2)} & z_{(1,2)} & z_{(0,3)} & z_{(2,2)} & z_{(1,3)} & z_{(0,4)} \end{bmatrix},$$

$$L_{x_1+x_2-1}^{(2)}(z) = \begin{bmatrix} z_{(1,0)} + z_{(0,1)} - z_{(0,0)} & z_{(2,0)} + z_{(1,1)} - z_{(1,0)} & z_{(1,1)} + z_{(0,2)} - z_{(0,1)} \\ z_{(2,0)} + z_{(1,1)} - z_{(1,0)} & z_{(3,0)} + z_{(2,1)} - z_{(2,0)} & z_{(2,1)} + z_{(1,2)} - z_{(1,1)} \\ z_{(1,1)} + z_{(0,2)} - z_{(0,1)} & z_{(2,1)} + z_{(1,2)} - z_{(1,1)} & z_{(1,2)} + z_{(0,3)} - z_{(0,2)} \end{bmatrix},$$

$$L_{x_1}^{(2)}(z) = \begin{bmatrix} z_{(1,0)} & z_{(2,0)} & z_{(1,1)} \\ z_{(2,0)} & z_{(3,0)} & z_{(2,1)} \\ z_{(1,1)} & z_{(2,1)} & z_{(1,2)} \end{bmatrix}, L_{x_2}^{(2)}(z) = \begin{bmatrix} z_{(0,1)} & z_{(1,1)} & z_{(0,2)} \\ z_{(1,1)} & z_{(2,1)} & z_{(1,2)} \\ z_{(0,2)} & z_{(1,2)} & z_{(0,3)} \end{bmatrix},$$

$$L_{1-x_1^2-x_2^2}^{(2)}(z) = \begin{bmatrix} z_{(0,0)} - z_{(2,0)} - z_{(0,2)} & z_{(1,0)} - z_{(3,0)} - z_{(1,2)} & z_{(0,1)} - z_{(2,1)} - z_{(0,3)} \\ z_{(1,0)} - z_{(3,0)} - z_{(1,2)} & z_{(2,0)} - z_{(4,0)} - z_{(2,2)} & z_{(1,1)} - z_{(3,1)} - z_{(1,3)} \\ z_{(0,1)} - z_{(2,1)} - z_{(0,3)} & z_{(1,1)} - z_{(3,1)} - z_{(1,3)} & z_{(0,2)} - z_{(2,2)} - z_{(0,4)} \end{bmatrix}.$$

It is easy to know that if $z \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ admits a Δ -measure μ , then

$$(2.6) \quad L_h^{(k)}(z) = 0, L_{g_j}^{(k)}(z) \succeq 0, \quad (j = 0, 1, \dots, n+1),$$

where $L_{g_j}^{(k)}(z) \succeq 0$ denotes that $L_{g_j}^{(k)}(z)$ is positive semidefinite. Hence (2.6) is necessary for z to admit a Δ -measure. However, the converse is not always true. We say z is *flat* if it satisfies not only (2.6) but also the following *rank condition*

$$(2.7) \quad \text{rank} M_{k-1}(z) = \text{rank} M_k(z).$$

Curto and Fialkow proved in [9] that if a tms $z \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ is flat, then z admits a unique Δ -measure, which is $\text{rank} M_k(z)$ -atomic.

Given two tms' $y \in \mathbb{R}^{\mathbb{N}_d^n}$ and $z \in \mathbb{R}^{\mathbb{N}_e^n}$, we say z is an *extension* of y , if $d \leq e$ and $y_\alpha = z_\alpha$ for all $\alpha \in \mathbb{N}_d^n$. In general, we denote by $z|_{\mathcal{E}}$ the subvector of z , whose entries are indexed by $\alpha \in \mathcal{E}$. For convenience, we denote by $z|_d$ the subvector $z|_{\mathbb{N}_d^n}$. If z is flat and extends y , we say z is a *flat extension* of y . Indeed, it is shown in [24, Proposition 3.3] that an \mathcal{E} -tms $\mathbf{a} \in \mathbb{R}^{\mathcal{E}}$ admits a Δ -measure if and only if it could be extendable to a flat tms $z \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ for some k . Hence based on (2.5), determining whether an E -matrix with an identifying vector $\mathbf{a} \in \mathbb{R}^{\mathcal{E}}$ is CP -completable could be transformed to investigating whether \mathbf{a} has a flat extension.

We now start to solve the CP -completion problem which has been formulated as the \mathcal{E} -T Δ MP. Let $d > \deg(\mathcal{E}) = 2$ and $R \in \mathbb{R}[x]_d$. Denote the coefficient vector of the polynomial R in the graded lexicographical ordering by $(R_\alpha)_{\alpha \in \mathbb{N}_d^n}$. Consider the following linear moment optimization problem

$$(2.8) \quad \begin{aligned} \min_z \quad & \sum_{\alpha \in \mathbb{N}_d^n} R_\alpha z_\alpha \\ \text{s.t.} \quad & z|_{\mathcal{E}} = \mathbf{a}, z \in \Upsilon_d(\Delta), \end{aligned}$$

where $\Upsilon_d(\Delta) = \{z \in \mathbb{R}^{\mathbb{N}_d^n} : \text{meas}(z, \Delta) \neq \emptyset\}$ with $\text{meas}(z, \Delta)$ being the set of all Δ -measures admitted by z .

Note that $\Delta \subseteq B(0, 1)$ is compact. If $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full, then the feasible set of (2.8) is compact convex and (2.8) has a minimizer for all R . If $\mathbb{R}[x]_{\mathcal{E}}$ is not Δ -full, it needs to choose R which is positive definite on Δ to guarantee that (2.8) has a minimizer. Therefore, to get a minimizer of (2.8), it is enough to solve (2.8) for a generic positive definite R , no matter whether $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full or not. Thus, we choose a generic $R \in \Sigma_{n,d}$, where $\Sigma_{n,d}$ is the set of all sum of squares polynomials in n variables with degree d .

Since $\Upsilon_d(\Delta)$ is quite difficult to describe, we relax it by (2.6) to the following set

$$(2.9) \quad \Gamma_k(h, g_B) := \left\{ z \in \mathbb{R}^{\mathbb{N}_{2k}^n} \mid L_h^{(k)}(z) = 0, L_{g_j}^{(k)}(z) \succeq 0, j = 0, 1, \dots, n+1 \right\},$$

where $k \geq d/2$ is an integer. Then the k -th order semidefinite relaxation of (2.8) can be stated as

$$(2.10) \quad (SDR)_k : \begin{array}{ll} \min_z & \sum_{\alpha \in \mathbb{N}_{2k}^n} R_\alpha z_\alpha \\ \text{s.t.} & z|_{\mathcal{E}} = \mathbf{a}, z \in \Gamma_k(h, g_B). \end{array}$$

Now we can use the following hierarchy SDP algorithm to solve the \mathcal{E} -T Δ MP. As mentioned above, we start with an even $d > \deg(\mathcal{E}) = 2$ and an order $k \geq d/2$.

Algorithm 2.1. The SDP algorithm for the \mathcal{E} -T Δ MP

Step 0: Choose a generic $R \in \Sigma_{n,d}$, and let $k := d/2$.

Step 1: Solve (2.10). If (2.10) is infeasible, then \mathbf{a} doesn't admit Δ -measures, and stop. Otherwise, compute a minimizer $z^{*,k}$. Let $t := 1$.

Step 2: Let $w := z^{*,k}|_{2t}$. If the rank condition (2.7) is not satisfied, go to Step 4.

Step 3: Compute the finitely atomic measure μ admitted by w :

$$\mu = \lambda_1 \delta(u_1) + \cdots + \lambda_m \delta(u_m),$$

where $m = \text{rank} M_t(w)$, $u_i \in \Delta$, $\lambda_i > 0$, and $\delta(u_i)$ is the Dirac measure supported on the point u_i ($i = 1, \dots, m$). Stop.

Step 4: If $t < k$, set $t := t + 1$ and go to Step 2; otherwise, set $k := k + 1$ and go to Step 1.

Remark 2.2. Algorithm 2.1 is a special case of Algorithm 4.2 in [24]. Denote $[x]_d := (x^\alpha)_{\alpha \in \mathbb{N}_d^n}$. We chose $R = [x]_{d/2}^T J^T J [x]_{d/2}$ in (2.10) where J is a random square matrix obeying Gaussian distribution. We checked the rank condition (2.7) numerically with the help of the singular value decomposition [17]. The rank of a matrix is evaluated as the number of its singular values which are greater than or equal to 10^{-6} . We used the method in [19] to get a m -atomic Δ -measure for w .

2.3. Properties of Algorithm 2.1. We first present some of the main results of Algorithm 2.1 as follows (cf. [24, Section 5]).

Theorem 2.3. 1) If \mathbf{a} doesn't admit Δ -measures and $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full, then $(SDR)_k$ is infeasible for all k big enough, which gives a certificate for the nonexistence of representing measures. 2) If \mathbf{a} admits a Δ -measure, then for almost all generated R : i) We could asymptotically get a flat extension of \mathbf{a} by solving the hierarchy $\{(SDR)_k\}_{k=1}^\infty$. ii) The obtained flat extension admits a m -atomic Δ -measure with $m \leq |\mathcal{E}|$, where $|\mathcal{E}|$ denotes the cardinality of the set \mathcal{E} .

Remark 2.4. Under some general conditions, which is almost sufficient and necessary, we could get a flat extension of \mathbf{a} by solving $(SDR)_k$ for some k [25–27], which implies that the SDP algorithm has finite convergence, and this always occur in our numerical experiments.

From the above, we know that if an E -matrix with the identifying vector $\mathbf{a} \in \mathbb{R}^{\mathcal{E}}$ could be completed to a CP -matrix, then we could asymptotically get a flat extension of \mathbf{a} for almost all $R \in \Sigma_{n,d}$ by Algorithm 2.1. Moreover, it could likely be obtained within finitely many steps. After getting a flat extension of \mathbf{a} , we could get a m -atomic Δ -measure for \mathbf{a} . Then we obtain a CP -completion matrix for the E -matrix, and a CP -decomposition is also produced.

Note that under the condition that $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full, Algorithm 2.1 could give a certificate for the nonexistence of CP -completeness. However, when $\mathbb{R}[x]_{\mathcal{E}}$ is not

Δ -full and \mathbf{a} doesn't admit Δ -measures, it is not clear whether there exists a k such that (2.10) is infeasible. So an interesting question is under which condition $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full? In the following, we will give a sufficient and necessary condition for $\mathbb{R}[x]_{\mathcal{E}}$ to be Δ -full.

Proposition 2.5. *Given $E = \{(i_1, j_1), \dots, (i_l, j_l) \mid 1 \leq i_k \leq j_k \leq n, k = 1, \dots, l\}$. Let \mathcal{E} be the index set of E and $\Delta = \{x \in \mathbb{R}_+^n : x_1 + \dots + x_n = 1\}$. Then $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full if and only if $\{(i, i), 1 \leq i \leq n\} \subseteq E$.*

Proof. We first prove the sufficient condition. If $\{(i, i), 1 \leq i \leq n\} \subseteq E$, then $\{(2, 0, \dots, 0), (0, 2, \dots, 0), \dots, (0, 0, \dots, 2)\} \subseteq \mathcal{E}$, so we have $x_i^2 \in \mathbb{R}[x]_{\mathcal{E}}$ for all $1 \leq i \leq n$. Hence for any $x \in \Delta$, there exists a polynomial $p(x) = \sum_{i=1}^n x_i^2 \in \mathbb{R}[x]_{\mathcal{E}}$ such that $p(x) > 0$, i.e. $p(x)|_{\Delta} > 0$. Thus $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full.

We prove the necessary condition by contradiction. Suppose there exists some $i_0 \in \{1, \dots, n\}$ such that $(i_0, i_0) \notin E$. Then for any polynomial $p(x) \in \mathbb{R}[x]_{\mathcal{E}}$, $p(x)$ is a linear combination of all the monomials of degree 2 except the monomial $x_{i_0}^2$. Let $c = (0, \dots, 0, 1_{i_0}, 0, \dots, 0)^T \in \Delta$ be a constant vector, then $p(c) = 0$ holds for all $p(x) \in \mathbb{R}[x]_{\mathcal{E}}$. Hence, there is no polynomial $p(x) \in \mathbb{R}[x]_{\mathcal{E}}$ such that $p(x)|_{\Delta} > 0$. This contradicts the fact that $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full. The proof is completed. \square

Remark 2.6. Proposition 2.5 shows that $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full if and only if all the diagonals of the E -matrix are given. So, when the diagonals are all given, Algorithm 2.1 could determine whether an E -matrix could be completed to a CP -matrix or not. Moreover, if it is CP -completable, not only a CP -completion but also a CP -decomposition could be obtained by Algorithm 2.1. In fact, Algorithm 2.1 computationally works even if there are ungiven diagonals in the E -matrix.

3. CP-COMPLETION WITH UNGIVEN DIAGONALS

In this section, we will study the CP -completion of the E -matrix under the other two cases. One is that all the diagonals of the E -matrix are ungiven, and the other is that the diagonals are partially given. Some properties of the E -matrix under these two cases will also be investigated.

3.1. All diagonals are ungiven. For a nonnegative E -matrix with all the diagonals ungiven, the following proposition shows that it can always be completed to a CP -matrix.

Proposition 3.1. *Suppose all the diagonals of the nonnegative E -matrix are ungiven. Then the E -matrix can always be completed to a CP -matrix.*

Proof. Without loss of generality, we suppose the E -matrix has the following form

$$(3.1) \quad \begin{bmatrix} * & A_{12} & \cdots & A_{1n} \\ A_{21} & * & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & * \end{bmatrix}_{n \times n},$$

where $A_{ij} (1 \leq i < j \leq n)$ are all given and nonnegative. Let

$$(3.2) \quad A = \sum_{1 \leq i < j \leq n} A_{ij} (\mathbf{e}_i + \mathbf{e}_j)(\mathbf{e}_i + \mathbf{e}_j)^T,$$

where \mathbf{e}_i denotes the i -th unit vector throughout the paper. It is easy to check that A is a CP -completion of (3.1). The proof is completed. \square

Remark 3.2. By Proposition 3.1, we know that the E -matrix with all diagonals ungiven is always CP completable. Algorithm 2.1 is guaranteed almost always to get a CP -completion if it exists. In our numerical experimentation, we always get it in finitely many steps. Moreover, it typically gives a CP -decomposition whose length is much smaller than the length in the proof.

3.2. Diagonals are partially given. In this subsection, we will study the CP -completion problem for the case that some of the diagonals of the E -matrix are given, while the other diagonals are ungiven. The submatrix that consists of all the rows and columns whose diagonals are given is called the *maximal principal submatrix* of the E -matrix. We have the following result.

Proposition 3.3. *Suppose the diagonals of the E -matrix are partially given. If the maximal principal submatrix of the E -matrix can not be completed to a CP -matrix, then the E -matrix can not be completed to a CP -matrix either.*

Proof. We prove by contradiction. Assume the E -matrix with the identifying vector $\mathbf{a} \in \mathbb{R}^{\mathcal{E}}$ is CP -completable. By (2.5), we know that \mathbf{a} admits a Δ -measure. Let the corresponding \mathcal{E}' -tms for the maximal principal submatrix be \mathbf{a}' . It is easy to know that \mathbf{a} is an extension of \mathbf{a}' without considering the orders of these two matrices. Since \mathbf{a} admits a Δ -measure μ , so does \mathbf{a}' . Then it follows from (2.5) that the maximal principal submatrix with the identifying vector \mathbf{a}' is also CP -completable, which leads to a contradiction. The proof is completed. \square

Remark 3.4. The converse of Proposition 3.3 is not necessarily true. For example, consider the nonnegative E -matrix

$$(3.3) \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 3 & * \end{bmatrix}.$$

Since the determinant of this matrix is equal to -1 for any nonnegative $*$ in $(1, 3)$ -th position, it is not positive semidefinite. So it can not be completed to a CP -matrix. However, its maximal principal submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is a CP -matrix.

Though the E -matrix (3.3) is not CP -completable, we could show that there always exists a sequence of CP -completable E -matrices whose identifying vectors converge to the identifying vector of the E -matrix. We first give the following definition.

Definition 3.5. A nonnegative E -matrix is called *partial CP -completable*, if its maximal principal submatrix is CP -completable.

Note that this definition is different from the definition of the partial CP -matrix in Section 1. As we know, if a matrix P is positive semidefinite, then $P + \epsilon I$ is positive definite for any $\epsilon > 0$, where I is the identity matrix. Hence we can regard the positive semidefinite matrix as a limit of some sequence of positive definite matrices. The converse is also true, i.e. the limit of any sequence of positive definite (or positive semidefinite) matrices is a positive semidefinite matrix. Fortunately, an analogous statement for partial CP -completable matrix also holds true. We show it in the following theorem.

Theorem 3.6. *Suppose the nonnegative E -matrix with the identifying vector $\mathbf{a} \in \mathbb{R}^{\mathcal{E}}$ is partial CP -completable. Then there exists a sequence of CP -completable E -matrices whose identifying vectors converge to \mathbf{a} , no matter whether the E -matrix is CP -completable or not.*

Proof. If the E -matrix is CP -completable, then there exists a CP -matrix A such that $\mathbf{a} = A|_E$. Choosing $A_k = A$ for all $k = 1, 2, \dots$, we know the theorem holds true.

For the case that the E -matrix is not CP -completable, without loss of generality, we assume it has the following form

$$(3.4) \quad \begin{bmatrix} & & & A_{1,n} \\ & A' & & \vdots \\ & & & A_{n-1,n} \\ A_{n,1} & \cdots & A_{n,n-1} & * \end{bmatrix}_{n \times n},$$

where the maximal principal submatrix A' is CP -completable and the off-diagonal entries in n -th row and column are all given and nonnegative. Consider the following sequence of E -matrices:

$$(3.5) \quad \begin{bmatrix} & & & A_{1,n} \\ & A' + \varepsilon_k I_{n-1} & & \vdots \\ & & & A_{n-1,n} \\ A_{n,1} & \cdots & A_{n,n-1} & * \end{bmatrix}_{n \times n}, \quad k = 1, 2, \dots,$$

where I_{n-1} is the identity matrix of order $n-1$ and $0 < \varepsilon_k \rightarrow 0$ as $k \rightarrow +\infty$. Hence the sequence of identifying vectors of (3.5) converges to \mathbf{a} as $k \rightarrow +\infty$. Let $\overline{A'}$ be a CP -completion matrix of A' , and

$$A_k = \begin{bmatrix} \overline{A'} & \mathbf{0} \\ \mathbf{0}^T & b \end{bmatrix}_{n \times n} + \sum_{1 \leq i \leq n-1} (\sqrt{\varepsilon_k} \mathbf{e}_i + \frac{A_{i,n}}{\sqrt{\varepsilon_k}} \mathbf{e}_n) (\sqrt{\varepsilon_k} \mathbf{e}_i + \frac{A_{i,n}}{\sqrt{\varepsilon_k}} \mathbf{e}_n)^T,$$

where $\mathbf{0}$ is the zero vector and b is any given nonnegative constant. Then A_k is a CP -completion of (3.5), which implies (3.5) is CP -completable. The proof is completed. \square

Remark 3.7. By Theorem 3.6, we know that if an E -matrix is partial CP -completable but not CP -completable, then it must be on the boundary of the set of CP -completable E -matrices, which implies that the set of CP -completable E -matrices is not closed.

Especially, when the E -matrix has only one diagonal given, we have the following result.

Proposition 3.8. *Suppose only one diagonal of the nonnegative E -matrix is given and positive. Then the E -matrix can always be completed to a CP -matrix.*

Proof. Without loss of generality, we suppose $n \geq 2$ and the E -matrix has the following form

$$(3.6) \quad \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & * & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & * \end{bmatrix}_{n \times n},$$

where $A_{11} > 0$ and all the off-diagonal entries $A_{ij} (i \neq j)$ are given and nonnegative. Let

$$\begin{aligned} A &= \sum_{1 \leq i < j \leq n} \left(\sqrt{\frac{A_{ii}}{n-1}} \mathbf{e}_i + \sqrt{\frac{n-1}{A_{ii}}} A_{ij} \mathbf{e}_j \right) \left(\sqrt{\frac{A_{ii}}{n-1}} \mathbf{e}_i + \sqrt{\frac{n-1}{A_{ii}}} A_{ij} \mathbf{e}_j \right)^T \\ &\quad + \sum_{2 \leq i < j \leq n} A_{ij} (\mathbf{e}_i + \mathbf{e}_j) (\mathbf{e}_i + \mathbf{e}_j)^T. \end{aligned}$$

Then, A is a CP -completion of (3.6). The proof is completed. \square

Remark 3.9. Note that: 1) If the E -matrix is CP -completable and its diagonals are partially given, we could still use Algorithm 2.1 to give a CP -decomposition whose length is relatively small. 2) If the E -matrix is not partial CP -completable, a certificate for the nonexistence (the (2.10) is infeasible) of CP -completion for the maximal principle submatrix could be obtained by applying Algorithm 2.1, which also implies the non- CP -completeness of the E -matrix by Proposition 3.3. 3) If the E -matrix is partial CP -completable, then Algorithm 2.1 is guaranteed almost always to get a CP -completion if it exists. In our numerical experimentation, we always get it in finitely many steps.

4. NUMERICAL EXPERIMENTS

In this section, we carried out some numerical experiments by using software GloptiPoly 3 [20] and SeDuMi [32]. We chose $d = 4$ and $k = 2$ in Step 0 of our Algorithm 2.1.

Example 4.1. Consider the nonnegative E -matrix with all the diagonals given (cf. [1, Exercise 2.57]):

$$(4.1) \quad \begin{bmatrix} b & 3 & 0 & * \\ 3 & 6 & 3 & 0 \\ 0 & 3 & 6 & 3 \\ * & 0 & 3 & b \end{bmatrix},$$

where b is a given nonnegative constant. It is easy to check that (4.1) is CP -completable if and only if $b \geq 3$. Without loss of generality, we take $b = 3$. Clearly, $E = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$ and is

$$\mathcal{E} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

We applied Algorithm 2.1 to the corresponding \mathcal{E} -tms $\mathbf{a} = (3, 3, 0, 6, 3, 0, 6, 3, 3)^T$, and obtained the following CP -completion

$$A = \begin{bmatrix} 3 & 3 & 0 & 0.0000 \\ 3 & 6 & 3 & 0 \\ 0 & 3 & 6 & 3 \\ 0.0000 & 0 & 3 & 3 \end{bmatrix} = \sum_{i=1}^3 \rho_i u_i u_i^T,$$

where the support points u_i and the corresponding Δ -representing measure weights ρ_i are given in Table 1. Thus (4.1) with $b = 3$ is CP -completable, and a CP -decomposition is also obtained as above.

i	u_i	ρ_i
1	$(0.5000, 0.5000, 0.0000, 0.0000)^T$	12.0000
2	$(0.0000, 0.0000, 0.5000, 0.5000)^T$	12.0000
3	$(0.0000, 0.5000, 0.5000, 0.0000)^T$	12.0000

TABLE 1. The points u_i and weights ρ_i for A in Example 4.1.

Example 4.2. Consider the nonnegative E -matrix with all the diagonals ungiven:

$$(4.2) \quad \begin{bmatrix} * & 4 & 1 & 2 & 2 \\ 4 & * & 0 & 1 & 3 \\ 1 & 0 & * & 1 & 2 \\ 2 & 1 & 1 & * & 1 \\ 2 & 3 & 2 & 1 & * \end{bmatrix}.$$

It follows from Proposition 3.1 that (4.2) can be completed to a CP -matrix. The completion matrix could be computed as (3.2). We also applied Algorithm 2.1 and obtained the following CP -completion matrix

$$A = \begin{bmatrix} 3.5379 & 4 & 1 & 2 & 2 \\ 4 & 8.4724 & 0 & 1 & 3 \\ 1 & 0 & 2.0113 & 1 & 2 \\ 2 & 1 & 1 & 1.5607 & 1 \\ 2 & 3 & 2 & 1 & 3.2138 \end{bmatrix} = \sum_{i=1}^5 \rho_i u_i u_i^T,$$

where u_i and ρ_i are shown in Table 2.

i	u_i	ρ_i
1	$(0.3318, 0.0000, 0.2080, 0.3222, 0.1380)^T$	11.6931
2	$(0.3077, 0.0000, 0.3198, 0.1908, 0.1819)^T$	1.9632
3	$(0.0000, 0.0000, 0.4420, 0.0327, 0.5253)^T$	6.6756
4	$(0.3711, 0.3459, 0.0000, 0.2329, 0.0499)^T$	3.8529
5	$(0.2315, 0.5291, 0.0000, 0.0455, 0.1937)^T$	28.6107

TABLE 2. The points u_i and weights ρ_i for A in Example 4.2.

We can see that Algorithm 2.1 could give a much shorter CP -decomposition than that given in (3.2).

Example 4.3. Consider the nonnegative E -matrix with diagonals partially given:

$$(4.3) \quad \begin{bmatrix} 6 & 4 & 1 & 2 & 2 \\ 4 & * & 0 & 1 & 3 \\ 1 & 0 & 3 & 1 & 2 \\ 2 & 1 & 1 & * & 1 \\ 2 & 3 & 2 & 1 & * \end{bmatrix}.$$

By Theorem 2.5, the corresponding $\mathbb{R}[x]_{\mathcal{E}}$ is not Δ -full in this example. However, we still could obtain the following CP -completion matrix by using Algorithm 2.1

$$A = \begin{bmatrix} 6 & 4 & 1 & 2 & 2 \\ 4 & 4.1241 & 0 & 1 & 3 \\ 1 & 0 & 3 & 1 & 2 \\ 2 & 1 & 1 & 1.0108 & 1 \\ 2 & 3 & 2 & 1 & 5.2354 \end{bmatrix} = \sum_{i=1}^5 \rho_i u_i u_i^T,$$

with u_i and ρ_i shown in Table 3.

i	u_i	ρ_i
1	$(0.4125, 0.0000, 0.3815, 0.2048, 0.0010)^T$	6.2038
2	$(0.0000, 0.0000, 0.4650, 0.1020, 0.4329)^T$	9.6328
3	$(0.4430, 0.3220, 0.0000, 0.1299, 0.1050)^T$	23.9046
4	$(0.1331, 0.3717, 0.0000, 0.0000, 0.4951)^T$	11.9082
5	$(0.1533, 0.0000, 0.0890, 0.3787, 0.3788)^T$	1.7209

TABLE 3. The points u_i and weights ρ_i for A in Example 4.3.

This also verifies the property of Algorithm 2.1 that if an E -matrix could be completed to a CP -matrix, no matter whether the corresponding $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full or not, a CP -completion could always be obtained by Algorithm 2.1.

Example 4.4. Consider the nonnegative E -matrix with one diagonal and all the off-diagonals given:

$$(4.4) \quad \begin{bmatrix} * & 7 & 1 & 3 & 9 \\ 7 & * & 5 & 8 & 5 \\ 1 & 5 & * & 2 & 2 \\ 3 & 8 & 2 & 3 & 1 \\ 9 & 5 & 2 & 1 & * \end{bmatrix}.$$

By Theorem 3.3, (4.4) could be completed to a CP -matrix. We can also apply Algorithm 2.1 to obtain a CP -completion and a CP -decomposition as follows:

$$A = \begin{bmatrix} 12.0587 & 7 & 1 & 3 & 9 \\ 7 & 26.8175 & 5 & 8 & 5 \\ 1 & 5 & 4.3096 & 2 & 2 \\ 3 & 8 & 2 & 3 & 1 \\ 9 & 5 & 2 & 1 & 11.5011 \end{bmatrix} = \sum_{i=1}^5 \rho_i u_i u_i^T,$$

where u_i and ρ_i are listed in Table 4.

Example 4.5. Consider the nonnegative E -matrix (cf. [1, Example 1.35]):

$$(4.5) \quad \begin{bmatrix} 1 & 1 & * & * & 0 \\ 1 & 1 & 1 & * & * \\ * & 1 & 1 & 1 & * \\ * & * & 1 & 1 & 1 \\ 0 & * & * & 1 & 1 \end{bmatrix}.$$

It was shown in [1] that (4.5) is not CP -completable. Since all the diagonals are given, we can use Algorithm 2.1 to verify this fact. Algorithm 2.1 terminates at

i	u_i	ρ_i
1	$(0.0017, 0.4932, 0.2021, 0.0988, 0.2042)^T$	48.4520
2	$(0.2505, 0.5393, 0.0062, 0.2040, 0.0000)^T$	50.8482
3	$(0.1796, 0.4880, 0.0000, 0.0917, 0.2407)^T$	1.0233
4	$(0.2154, 0.0000, 0.5541, 0.2305, 0.0000)^T$	7.5805
5	$(0.4869, 0.0000, 0.0000, 0.0000, 0.5131)^T$	35.7825

TABLE 4. The points u_i and weights ρ_i for A in Example 4.4.

Step 2 with $k = 3$ as (2.10) is infeasible, which indicates that (4.5) is not CP -completable.

Example 4.6. Consider the nonnegative E -matrix with the diagonals partially given:

$$(4.6) \quad \begin{bmatrix} 1 & 1 & 2 & * & 4 \\ 1 & 1 & 3 & * & 3 \\ 2 & 3 & 3 & 3 & * \\ * & * & 3 & 2 & * \\ 4 & 3 & * & * & * \end{bmatrix}.$$

The maximal principle submatrix of (4.6) is

$$(4.7) \quad \begin{bmatrix} 1 & 1 & 2 & * \\ 1 & 1 & 3 & * \\ 2 & 3 & 3 & 3 \\ * & * & 3 & 2 \end{bmatrix}.$$

Since all the diagonals of (4.7) are given, we can use Algorithm 2.1 to verify whether (4.7) is CP -completable or not. Algorithm 2.1 terminates at Step 2 with $k = 2$, as (2.10) is infeasible. Thus (4.7) is not CP -completable, which indicates that (4.6) is not CP -completable either by Proposition 3.3.

Example 4.7. Consider the nonnegative E -matrix (3.3) which was mentioned in subsection 3.2. We rewrite it as follows:

$$(4.8) \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 3 & * \end{bmatrix}.$$

Though its maximal principal submatrix

$$A' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is CP -completable, (4.8) is not. We considered the approximations of (4.8) in the form (3.5). Choosing $\varepsilon_k = 10^{-k}$, we have the E -matrix sequence

$$(4.9) \quad \begin{bmatrix} 1 + 10^{-k} & 1 & 2 \\ 1 & 1 + 10^{-k} & 3 \\ 2 & 3 & * \end{bmatrix}, \quad k = 1, 2, \dots$$

By the proof of Theorem 3.6, we know (4.9) is CP -completable for all k . We used Algorithm 2.1 and obtained the following CP -completion matrices

$$A_1 = \begin{bmatrix} 1.1 & 1 & 2 \\ 1 & 1.1 & 3 \\ 2 & 3 & 10.9524 \end{bmatrix} = \sum_{i=1}^2 \lambda_i u_i u_i^T,$$

$$A_2 = \begin{bmatrix} 1.01 & 1 & 2 \\ 1 & 1.01 & 3 \\ 2 & 3 & 56.2189 \end{bmatrix} = \sum_{i=1}^2 \rho_i v_i v_i^T,$$

and

$$A_3 = \begin{bmatrix} 1.001 & 1 & 2 \\ 1 & 1.001 & 3 \\ 2 & 3 & 487.2967 \end{bmatrix} = \sum_{i=1}^4 \sigma_i \omega_i \omega_i^T,$$

for $k = 1, 2$ and 3 , respectively. The support points and the corresponding Δ -representing measure weights are shown in Table 5 and 6.

i	u_i	λ_i	v_i	ρ_i
1	$(0.1254, 0.1881, 0.6866)^T$	23.2350	$(0.0327, 0.0490, 0.9183)^T$	66.6636
2	$(0.6190, 0.3810, 0.0000)^T$	1.9174	$(0.5124, 0.4876, 0.0000)^T$	3.5753

TABLE 5. The points and weights for A_1 and A_2 in Example 4.7.

i	ω_i	σ_i
1	$(0.5012, 0.4987, 0.0001)^T$	3.7527
2	$(0.0682, 0.0696, 0.8623)^T$	11.6502
3	$(0.0027, 0.0048, 0.9925)^T$	485.9000
4	$(0.4545, 0.3039, 0.2487)^T$	0.0023

TABLE 6. The support points and the weights for A_3 in Example 4.7.

Remark 4.8. For the case that the E -matrix have diagonals ungiven, if the maximal principle submatrix is not CP -completable, then the E -matrix is not either. But even if the maximal principle submatrix is CP -completable, the E -matrix may not be CP -completable. Anyway, we know from Theorem 3.6 and Remark 3.7 that the E -matrix is in the closure of CP -completable E -matrices; there always exists a sequence of CP -completable E -matrices with the identifying vectors converging to that of the E -matrix. Moreover, Algorithm 2.1 could give not only a CP -completion but also a CP -factorization of the approximate CP -matrix.

5. CONCLUSION

In this paper, we have successfully solved the CP -completion problem for the nonnegative E -matrix. We formulate it as an \mathcal{E} -T Δ MP and present an SDP Algorithm 2.1 for it. For the case that all the diagonal entries are given, since the corresponding $\mathbb{R}[x]_{\mathcal{E}}$ is Δ -full, Algorithm 2.1 can give a certificate for the non- CP -completeness of the E -matrix. Moreover, if the E -matrix is CP -completable, Algorithm 2.1 is almost always guaranteed to get not only a CP -completion but

also a CP -decomposition. For the E -matrix with all diagonals ungiven, we show it can always be completed to a CP -matrix and Algorithm 2.1 can almost always obtain a CP -completion and a CP -decomposition with a smaller length. For the E -matrix with diagonals partially given, if it is not partial CP -completable, then a certificate can be obtained; if it is partial CP -completable, then there always exists a sequence of CP -completable E -matrices whose identifying vector converges to the identifying vector of the E -matrix, Algorithm 2.1 is almost always guaranteed to get a CP -completion if it is CP -completable. In particular, if the E -matrix has only one positive diagonal given, it is always CP -completable.

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DEPARTMENT OF MATHEMATICS, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI 200240, P.R. CHINA

E-mail address: `congcongyan@sjtu.edu.cn`

DEPARTMENT OF MATHEMATICS, AND MOE-LSC, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI 200240, P.R. CHINA

E-mail address: `jyfan@sjtu.edu.cn`